SOLITONS OF DISCRETE CURVE SHORTENING

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ABSTRACT. For a polygon $x = (x_j)_{j\in\mathbb{Z}}$ in $\mathbb{R}^n$ we consider the polygon $(T(x))_j = \{x_{j-1} + 2x_j + x_{j+1}\}/4$. This transformation is obtained by applying the midpoints polygon construction twice. For a closed polygon or a polygon with finite vertices this is a curve shortening process. We call a polygon x a *soliton* of the transformation T if the polygon $T(x)$ is an affine image of x. We describe a large class of solitons for the transformation T by considering smooth curves c which are solutions of the differential equation $\ddot{c}(t) = Bc(t) + d$ for a real matrix B and a vector d. The solutions of this differential equation can be written in terms of power series in the matrix B. For a solution c and for any $s > 0, a \in \mathbb{R}$ the polygon $x(a,s) = (x_j(a,s)_j)_{j\in\mathbb{Z}}; x_j(a,s) = c(a + sj)$ is a soliton of T. For example we obtain solitons lying on spiral curves which under the transformation T rotate and shrink.

1. INTRODUCTION

For an infinite polygon $(x_j)_{j\in\mathbb{Z}}$ given by the vertices x_j in the vector space $\mathbb{R}^n$ the *midpoints polygon* is defined by $M(x)$ where $M(x)_j := (x_j + x_{j+1})/2; j \in \mathbb{Z}$. If the polygon is closed or rather periodic, i.e. if for some $N : x_{j+N} = x_j$ for all $j \in \mathbb{Z}$ then this midpoint mapping M defines a *curve shortening process,* i.e. the polygon $M(x)$ is shorter than the polygon x unless it is a single point. If we iterate the process for a closed polygon $(x_j)_{j=1,...,N}$ it converges to the barycenter $(x_1 + \ldots + x_N)/N$. This elementary construction was already used by Darboux in 1878. He also showed in [6] that in the plane in the general case the sequence $M^k(x)/\cos(\pi/N)^k$ converges for $k \to \infty$ to an ellipse. These results for polygons in Euclidean space were later rediscovered and extended by several authors, for example Kasner [10], Schoeneberg [14], and Berlekamp et al. [2]. Bruckstein & Shaked [3] discuss the relation with iterative smoothing procedures in shape analysis and recognition. Nowadays applications of discrete curve shortening are discussed in several papers. For example Smith et al. [15] present its connection with the rendezvous problem for mobile autonomous robots.

We modify the curve shortening process M for an infinite polygon (i.e. not necessarily closed) in Euclidean space as follows: Instead of the midpoint

mapping M we apply the midpoint mapping twice and use an index shift, i.e. we define the polygon $T(x) = (T(x)_j)_{j \in \mathbb{Z}}$ by the Equation

$$(1) \qquad T(x)_j = \frac{1}{4} \{x_{j-1} + 2x_j + x_{j+1}\} \ ,$$

i.e. $T(x)_j = (M^2(x))_{j-1}$. Introducing the index shift has the following advantage: Since

$$(2) \qquad T(x)_j - x_j = \frac{1}{4} \{x_{j-1} - 2x_j + x_{j+1}\}$$

we can view this process T as a discrete approximation of the *semidiscrete flow* $s \mapsto x_j(s)$ defined by

$$(3) \qquad \frac{dx_j(s)}{ds} = x_{j-1}(s) - 2x_j(s) + x_{j+1}(s), \ x_j(0) = x_j$$

on the space of polygons. On the space of closed polygons this flow is the negative gradient flow of the functional F_2, cf. Section 4. Semidiscrete flows are discussed for example by Chow& Glickenstein [5].

The polygon $T(x)$ is formed by the midpoints $T(x)_j$ of the medians through x_j of the triangle formed by x_{j-1}, x_j, x_{j+1}. Linear polygons $x_j = ju + v$; $u, v \in \mathbb{R}^n$ are the fixed points of T.

Since the process T is affinely invariant it is natural to consider polygons which are mapped under T onto an affine image of themselves. We also use the term *soliton* for these polygons in analogy to the case of smooth curves or manifolds which are mapped under the mean curvature flow onto the same curve or rather manifold up to an isometry, cf. Hungerbühler & Smoczyk [9], Hungerbühler & Roost [8] and Altschuler [1]. We call a polygon $x = (x_j)_j$ a *soliton* of the process T if there exists an affine mapping $x \in \mathbb{R}^n \mapsto Ax + b \in \mathbb{R}^n$ for a matrix A and a vector b such that

$$(4) \qquad T(x)_j = \frac{1}{4} \{x_{j-1} + 2x_j + x_{j+1}\} = Ax_j + b$$

for all $j \in \mathbb{Z}$. The main idea of this paper is to consider not only polygons, but also smooth curves c which are affinely invariant under an analogous process on curves: We adapt the process T to curves in the following sense: We associate the one-parameter family $c_s, s \in \mathbb{R}$ to the smooth curve c :

$$(5) \qquad c_s(t) := \frac{1}{4} \{c(t-s) + 2c(t) + c(t+s)\} \ .$$

For $s > 0, a \in \mathbb{R}$ we define the polygon $x(a, s) = (x_j(a, s))_{j \in \mathbb{Z}}; x_j(a, s) = c(a + sj)$. Then the smooth curve c defines solitons of the form $x(a, s)$ for the curve shortening process T if the following holds: For some $\epsilon > 0$ and any $s \in (0, \epsilon)$ there is a one-parameter family of affine maps $x \in \mathbb{R}^n \mapsto A(s)x + b(s) \in \mathbb{R}^n$ such that for any $s \in (0, \epsilon)$ and all $t \in \mathbb{R}$:

$$(*) \qquad c_s(t) = \frac{1}{4} \{c(t-s) + 2c(t) + c(t+s)\} = A(s)c(t) + b(s) \ .$$

In Figure 1 we show a polygon $x = (x_j)$ which is a soliton of the pro-
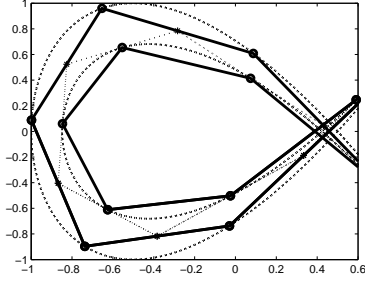
FIGURE 1. polygon
as soliton of T
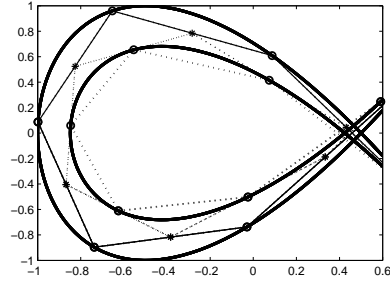
FIGURE 2. smooth
curve as soliton of T

cess T. Its vertices $x_j = c\,(0.4 \cdot j)\,, j \in \mathbb{Z}$ lie on the smooth curve $c(t) = (\cos(2t), \cos(3t))$, which is also a soliton of the process T, see Figure 2. Denote by A the diagonal matrix with entries $(1 + \cos(0.8))/2$ and $(1 + \cos(1.2))/2$. Then in Figure 1 the polyon x and its image $T(x)$ are shown, which satisfy Equation (4). In Figure 2 the smooth curves c and $c_{0.4}$ are shown, which satisfy Equation ($*$) for $s = 0.4, A(0.4) = A, b(0.4) = 0$. Hence the process T in this case corresponds to a *scaling.* This example belongs to Case (1a) in Section 5. Here $A(0.4)$ is given by Equation (8) where B is the diagonal matrix with entries $b_1 = -4$ and $b_2 = -9$.

It is the main result of this paper that solutions of Equation ($*$), i.e. smooth curves c defining solitons for the curve shortening process T, can be characterized as solutions of the inhomogeneous linear differential equation

$$(6) \qquad\qquad \ddot{c} = Bc(t) + d$$

of second order with constant coefficients with $B = 2\,A''(0), d = 2\,b''(0)$, cf. Theorem 1 and Theorem 2. The solitons c and the maps $A(s)$ or rather $b(s)$ can be described in terms of the power series

$$(7) \qquad \mathrm{co}_B(t) := \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} B^k \ , \ \ \mathrm{si}_B(t) := \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} B^k \,,$$

cf. Proposition 2 and Proposition 3 in Section 2 .
We will show, that for any real matrix B solutions of Equation (6) are solitons of the curve shortening process and the matrix $A(s)$ of Equation ($*$) is given by

$$(8) \qquad\qquad A(s) = \frac{1}{2} \left\{ \mathbb{1} + \mathrm{co}_B(s) \right\} \,.$$

The vectors $b(s)$ depend on the structure of the matrix B and vanish for closed solitons.
The one-parameter family $s \mapsto c_s(t)$ defined by Equation (5) associated to a soliton c can be used to define an affinely invariant solution of the *wave equation,* cf. Remark 3. The relation of polygonal curve shortening and the

curve shortening flow for smooth curves is discussed in many papers, cf. [13, Sec.3], a reference for curve shortening flows is the book [4]. Self-similar solutions of the Euclidean curve shortening flow in the plane, i.e. the mean curvature flow for curves in Euclidean space, are discussed by Halldorsson [7], see also Hungerbühler & Smoczyk [9] and Altschuler [1]. These questions lead to systems of non-linear ordinary differential equations.

In Section 4 we show that the solitons of the curve shortening T are also solitons of the *semidiscrete flow* defined in Equation (3). In Section 5 we discuss the planar case $n = 2$ in detail and present examples. For example the *parabola* is a soliton for which the curve shortening leads to a *translation.* We also obtain various spirals as solitons which rotate and shrink and closed curves of Lissajous type, which scale under the mapping T. The results of Section 5 should be compared with the *zoo of solitons* obtained for the Euclidean curve shortening flow by Halldorsson [7], see also Hungerbühler & Smoczyk [9] and Altschuler [1].

2. System of linear differential equations of second order

Let $B \in M(n; \mathbb{R})$ be a $n \times n$ matrix with real entries. We denote by $\mathbb{1}$ the identity matrix and use the following notation:

Using the *matrix exponential* $\exp(B) := \sum_{k=0}^{\infty} B^k/k!$ one can also define the matrix functions $\cosh(B), \cos(B), \sin(B), \sinh(B)$ as power series in B. For commuting matrices B, C (i.e. $BC = CB$) we have

$$\exp(B + C) = \exp(B) \exp(C) = \exp(C) \exp(B) ,$$

cf. [12, ch.6.1].

Definition 1. *For a matrix $B \in M(n, \mathbb{R})$ we define the following mappings* $\mathrm{co}_B, \mathrm{si}_B : \mathbb{R} \to M(n, \mathbb{R}) :$

$$\mathrm{co}_B(t) \quad := \quad \sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!} B^k = \mathbb{1} + \frac{t^2}{2!} B + \frac{t^4}{4!} B^2 + \dots$$

$$\mathrm{si}_B(t) \quad := \quad \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!} B^k = t\mathbb{1} + \frac{t^3}{3!} B + \frac{t^5}{5!} B^2 + \dots$$

These power series are obviously convergent and we obtain the following

Proposition 1 (Properties of $\mathrm{co}_B(t), \mathrm{si}_B(t)$). *For a matrix B the above defined mappings $\mathrm{co}_B(t), \mathrm{si}_B(t)$ satisfy the following (differential) equations:*

$$\frac{d}{dt} \mathrm{co}_B(t) = \mathrm{co}'_B(t) = B \cdot \mathrm{si}_B(t) \,;\, \mathrm{si}'_B(t) = \mathrm{co}_B(t)$$

$$\mathrm{co}''_B(t) = B \cdot \mathrm{co}_B(t) \,;\, \mathrm{si}''_B(t) = B \cdot \mathrm{si}_B(t)$$

$$\mathrm{co}_B(t)^2 - B \cdot \mathrm{si}_B(t)^2 = \mathbb{1} \,.$$

Given a real number $b \in \mathbb{R}$ we denote by $\cos_b, \sin_b : \mathbb{R} \to \mathbb{R}$ the unique solutions of the differential equation $f'' = bf$ with $\cos_b(0) = 1, \cos'_b(0) =$

$0, \sin_b(0) = 0, \sin_b'(0) = 1$. Hence

(9)
$$\cos_b(t) = \begin{cases} \cosh\left(\sqrt{b}t\right) & ; \quad b > 0 \\ 1 & ; \quad b = 0 \\ \cos\left(\sqrt{-b}t\right) & ; \quad b < 0 \end{cases} ; \sin_b(t) = \begin{cases} \sinh\left(\sqrt{b}t\right)/\sqrt{b} & ; \quad b > 0 \\ t & ; \quad b = 0 \\ \sin\left(\sqrt{-b}t\right)/\sqrt{-b} & ; \quad b < 0 \end{cases}$$

Then we obtain for a real number b and the matrix $B = b\mathbb{1}$:

$$\mathrm{co}_B(t) = \mathrm{co}_{b\mathbb{1}}(t) = \cos_b(t)\mathbb{1} \, ; \, \mathrm{si}_B(t) = \mathrm{si}_{b\mathbb{1}}(t) = \sin_b(t)\mathbb{1} \, .$$

If $B = C^2, C \in M(n, \mathbb{C})$ then $\mathrm{co}_B(t) = \cosh(Ct), C \cdot \mathrm{si}_B(t) = \sinh(tC)$. For $B = -C^2, C \in M(n, \mathbb{C})$ we obtain $\mathrm{co}_B(t) = \cos(Ct), C \cdot \mathrm{si}_B(t) = \sin(Ct)$.

Proposition 2 (HOMOGENEOUS DIFFERENTIAL EQUATION). *For a matrix $B \in M(n; \mathbb{R})$ and vectors $v, w \in \mathbb{R}^n$ the linear system of ordinary differential equations (with constant coefficients) of second order:*

(10)
$$\ddot{c}(t) = Bc(t)$$

with initial values $c(0) = v, \dot{c}(0) = w$ has the unique solution

(11)
$$c(t) = c[v, w](t) := \mathrm{co}_B(t)(v) + \mathrm{si}_B(t)(w) \, .$$

Then the one-parameter family of curves $s \mapsto c_s$ for $s \in \mathbb{R}$ defined by Equation (5) satisfies

(12)
$$c_s(t) = A(s) \, c(t) \, ,$$

with

$$A(s) = \frac{1}{2} \left\{ \mathbb{1} + \mathrm{co}_B(s) \right\}$$

i.e. c_s is a linear image of c for any s and Equations (*) and Equations (8) hold with $b(s) = 0$.

Proof. From Proposition 1 it follows that $c(t) = c[v, w](t)$ is a solution of Equation (10) with $c(0) = v, \ddot{c}(0) = w$. Equation (5) implies that c_s is the unique solution of Equation (10) with initial conditions:

$$\begin{aligned} c_s(0) &= \frac{1}{4} \left\{ c(-s) + 2c(0) + c(s) \right\} = \frac{1}{2} \left\{ \mathbb{1} + \mathrm{co}_B(s) \right\} (v) \\ \dot{c}_s(0) &= \frac{1}{4} \left\{ \dot{c}(-s) + 2\dot{c}(0) + \dot{c}(s) \right\} = \frac{1}{2} \left\{ \mathbb{1} + \mathrm{co}_B(s) \right\} (w) \, . \end{aligned}$$

Using Equation (11) we can write

$$\begin{aligned} c_s(t) &= \frac{1}{2} \mathrm{co}_B(t) \left\{ \mathbb{1} + \mathrm{co}_B(s) \right\} (v) + \mathrm{si}_B(t) \left\{ \mathbb{1} + \mathrm{co}_B(s) \right\} (w) \\ &= \frac{1}{2} \left\{ \mathbb{1} + \mathrm{co}_B(s) \right\} \left\{ \mathrm{co}_B(t)(v) + \mathrm{si}_B(t)(w) \right\} = \frac{1}{2} \left\{ \mathbb{1} + \mathrm{co}_B(s) \right\} (c(t)) \, . \end{aligned}$$

$\square$

Remark 1 (ADDITION RULES). The second order linear system (10) of ordinary differential equations is equivalent to the following first order linear system of ordinary differential equations

$$\frac{d}{dt}\begin{pmatrix} c(t) \\ \dot{c}(t) \end{pmatrix} = \begin{pmatrix} 0 & \mathbb{1} \\ B & 0 \end{pmatrix}\begin{pmatrix} c(t) \\ \dot{c}(t) \end{pmatrix}$$

with constant coefficients. The solution of this system with initial values $c(0) = v, \dot{c}(0) = w$ is given by

$$\begin{pmatrix} c(t) \\ \dot{c}(t) \end{pmatrix} = \exp\left(\begin{pmatrix} 0 & \mathbb{1} \\ B & 0 \end{pmatrix}t\right)\begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \mathrm{co}_B(t) & \mathrm{si}_B(t) \\ B\cdot\mathrm{si}_B(t) & \mathrm{co}_B(t) \end{pmatrix}\begin{pmatrix} v \\ w \end{pmatrix}.$$

Hence the curve $t \mapsto (c(t), \dot{c}(t))$ is the orbit of the point $(v, w) \in \mathbb{R}^n \oplus \mathbb{R}^n$ under a one-parameter group of linear transformations.
Since

$$\exp\left(\begin{pmatrix} 0 & \mathbb{1} \\ B & 0 \end{pmatrix}(t+s)\right) = \begin{pmatrix} \mathrm{co}_B(t+s) & \mathrm{si}_B(t+s) \\ B\cdot\mathrm{si}_B(t+s) & \mathrm{co}_B(t+s) \end{pmatrix}$$

$$= \exp\left(\begin{pmatrix} 0 & \mathbb{1} \\ B & 0 \end{pmatrix}t\right)\cdot\exp\left(\begin{pmatrix} 0 & \mathbb{1} \\ B & 0 \end{pmatrix}s\right)$$

$$= \begin{pmatrix} \mathrm{co}_B(t) & \mathrm{si}_B(t) \\ B\cdot\mathrm{si}_B(t) & \mathrm{co}_B(t) \end{pmatrix}\cdot\begin{pmatrix} \mathrm{co}_B(s) & \mathrm{si}_B(s) \\ B\cdot\mathrm{si}_B(s) & \mathrm{co}_B(s) \end{pmatrix}$$

we obtain the following addition rules:

$$(13) \qquad \mathrm{co}_B(t+s) = \mathrm{co}_B(t)\cdot\mathrm{co}_B(s) + B\cdot\mathrm{si}_B(t)\cdot\mathrm{si}_B(s)$$

$$(14) \qquad \mathrm{si}_B(t+s) = \mathrm{co}_B(t)\cdot\mathrm{si}_B(s) + \mathrm{si}_B(t)\cdot\mathrm{co}_B(s).$$

Remark 2 (ROOTS OF B). If there is a matrix $C \in M(n; \mathbb{R})$, such that $B = C^2$ then the curve $c(t) = c[v, Cv](t) = \exp(tC)(v)$ satisfies $c(t+s) = \exp(sC)\,c(t)$, i.e. the curve is the orbit of a point under a one-parameter group of affine transformations in $\mathbb{R}^n$.

Let c be a solution of the inhomogeneous linear differential equation $\ddot{c}(t) = Bc(t) + d$, let $U \in M(n; \mathbb{R})$ be an invertible matrix and e a vector, then $f(t) = Uc(t) + e$ solves the differential equation $\ddot{f}(t) = UBU^{-1}f(t) + \left(\mathbb{1} - UBU^{-1}\right)e + Ud$. Therefore one can assume that $B \in M(n; \mathbb{R})$ has already complex Jordan normal form cf. [11, Thm. 5.4.10] or real Jordan normal form, cf. [11, Thm. 5.6.3]. If an eigenvalue λ of A is not real then the conjugate value $\overline{\lambda}$ is an eigenvalue, too. In Section 5 we discuss the possible real Jordan normal forms for $n = 2$. For a complex number λ the Jordan block $J_m(\lambda)$ is given by:

$$(15) \qquad J_m(\lambda) := \left.\begin{pmatrix} \lambda & 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ \dots & & & & & \\ \dots & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & 0 & \dots & \lambda \end{pmatrix}\right\} m\,.$$

The *complex Jordan normal form* of a real matrix B consists of Jordan blocks of the form $J_m(\lambda), m \geq 1$ for an eigenvalue $\lambda \in \mathbb{C}$.

The *real Jordan normal form* of a real matrix B consists of two different types of Jordan blocks: For a real eigenvalue λ there are Jordan blocks of the form $J_m(\lambda), m \geq 1$. If $\mu = \alpha + i\beta, \alpha \in \mathbb{R}, \beta \in \mathbb{R} \setminus \{0\}$ is a non-real eigenvalue, then for some $m \geq 1$ real Jordan blocks of the form

$$(16) \qquad J_{2m}(\alpha, \beta) := \left. \left(\begin{array}{cc} J_m(\alpha) & -\beta\mathbb{1} \\ \beta\mathbb{1} & J_m(\alpha) \end{array} \right) \right\} 2m \,,$$

occur. The real Jordan block $J_{2m}(\alpha, \beta)$ is always invertible. Hence the Jordan normal form of a singular real matrix contains a nilpotent Jordan block of the form $J_m(0), m \geq 1$. Therefore it is sufficient to discuss the following cases:

Proposition 3 (INHOMOGENEOUS DIFFERENTIAL EQUATION). *For a matrix $B \in M(n; \mathbb{R})$ and a vector $d \in \mathbb{R}^n$ we consider the inhomogeneous linear system of ordinary differential equations (with constant coefficients) of second order:*

$$(17) \qquad \ddot{c}(t) = Bc(t) + d \,,$$

and we consider the one-parameter family of curves c_s defined by Equation (5). Then we have

$$(18) \qquad c_s(t) = A(s)\, c(t) + b(s) \,,$$

with

$$A(s) = \frac{1}{2} \left\{ \mathbb{1} + \mathrm{co}_B(s) \right\}$$

i.e. Equation $(*)$ and Equation (8) hold .

We consider three cases:

(a) *If there is a vector d_* such that $d = B \cdot d_*$, then for $v, w \in \mathbb{R}^n$ the unique solution of Equation (17) with intitial values $c(0) = v, \dot{c}(0) = w$ is given by*

$$(19) \qquad c(t) = c[v, w](t) = \mathrm{co}_B(t)\, (v + d_*) + \mathrm{si}_B(t)(w) - d_* \,.$$

   and

$$b(s) = \frac{1}{2} \left\{ \mathrm{co}_B(s) - \mathbb{1} \right\} (d_*)$$

(b) *Let B be the nilpotent Jordan block matrix: $B = N_n := J_n(0)$; cf. Equation 15. Hence $N_n(e_1) = 0$ and $N_n(e_j) = e_{j-1}$ for $j \geq 2$ for a basis $e_1, e_2, \ldots, e_n$ of $\mathbb{R}^n$. Then the unique solution $c_* = c_*(t)$ of*

$$(20) \qquad \ddot{c}_*(t) = N_n c_*(t) + e_n, c_*(0) = \dot{c}_*(0) = 0$$

   is given by

$$c_*(t) = \left(\frac{t^{2n}}{(2n)!}, \ldots, \frac{t^2}{2!} \right).$$

$d \in \mathbb{R}^n$ *can be written as* $d = (d_1, d_2, \ldots, d_n) = B \cdot d_* + d_n e_n$ *with* $d_* = (0, d_1, d_2, \ldots, d_{n-1})$. *And the unique solution of Equation* (17) *with intitial values* $c(0) = v, \dot{c}(0) = w$ *is given by*

$$(21) \qquad c(t) = \mathrm{co}_B(t)\,(v + d_*) + \mathrm{si}_B(t)(w) - d_* + d_n c_*(t)$$

$$(22) \qquad b(s) = \frac{1}{2}\left\{\mathrm{co}_B(s) - \mathbb{1}\right\}(d_*) + \frac{1}{2}\, d_n c_*(s)\,.$$

(c) *If* $B = 0$ *then the unique solution of Equation* (17) *with initial values* $v = c(0), w = \dot{c}(0)$ *and the one-parameter family* $c_s(t)$ *is given by:*

$$c(t) = c[v,w](t) = \frac{t^2}{2}\,d + wt + v\,;\; c_s(t) = c(t) + \frac{1}{4}\, s^2 d$$

and

$$b(s) = \frac{1}{4}\, s^2 d\,.$$

Proof. The curve

$$c(t) = c[v,w](t) = \mathrm{co}_B(t)\,(v + d_*) + \mathrm{si}_B(t)(w) - d_*$$

with $c(0) = v, \dot{c}(0) = w$ satisfies by Proposition 1:

$$\ddot{c}(t) = B\left\{\mathrm{co}_B(t)\,(v + d_*) + \mathrm{si}_B(t)(w)\right\} = B\,(c(t) + d_*) = Bc(t) + B(d_*)\,.$$

This already proves Equation (17) in case (a), or rather Equation (19). The addition rules Equation (13) show:

$$c_s(t) = \frac{1}{4}\left\{c(t-s) + 2c(t) + c(t+s)\right\}$$

$$= \frac{1}{4}\,\left[\mathrm{co}_B(t-s) + \mathrm{co}_B(t+s)\right](v + d_*) + \frac{1}{2}\,\mathrm{co}_B(t)\,(v + d_*) +$$

$$\frac{1}{4}\,\left[\mathrm{si}_B(t-s) + \mathrm{si}_B(t+s)\right](w) + \frac{1}{2}\,\mathrm{si}_B(t)(w) - d_*$$

$$(23) \qquad = \frac{1}{2}\,\left\{\mathbb{1} + \mathrm{co}_B(s)\right\}\left[\mathrm{co}_B(t)(v + d_*) + \mathrm{si}_B(t)(w)\right] - d_*$$

$$= A(s)(c(t)) + b(s)$$

with

$$A(s) = \frac{1}{2}\left\{\mathrm{co}_B(s) + \mathbb{1}\right\}\,;\; b(s) = \frac{1}{2}\left\{\mathrm{co}_B(s) - \mathbb{1}\right\}(d_*)\,.$$

(b) Now let $B = N_n$. One checks that $c_*(t)$ given by Equation (20) is the unique solution of Equation (17) with $c(0) = \dot{c}(0) = 0$ and $b = e_n$. Then for any solution $c = c(t)$ of Equation (17) the curve $c(t) - d_n c_*(t)$ is a solution of Equation

$$\ddot{e}(t) = B\,(e(t) + d_*) = Be(t) + (d_1, \ldots, d_{n-1}, 0)\,.$$

Hence we conclude from Part (a) that

$$e(t) = \mathrm{co}_B(t)\,(v + d_*) + \mathrm{si}_B(t)(w) - d_*.$$

This implies that $c(t)$ satisfies Equation (21). Since

$$\frac{1}{4}\left\{c_*(t-s)+2c_*(t)+c_*(t+s)\right\}=\frac{1}{2}\left\{\text{co}_B(s)+\mathbb{1}\right\}(c_*(t))$$

together with Equation (23) proves Equation ($*$) and Equation (22).

(c) For $B=0$ Equation (17) is $\ddot{c}(t)=d$, hence we obtain $c(t)=dt^2/2+wt+v$. Furthermore $\text{co}_B(s)=\mathbb{1}=A(s)$ and $c_s(t)=\frac{1}{2}\left\{c(t-s)+2c(t)+c(t+s)\right\}=c(t)+\frac{1}{4}s^2d$. $\qquad\square$

Remark 3 (WAVE EQUATION). Let $c=c(t)$ be a smooth curve with the one-parameter family of curves c_s defined by Equation (5). Then

$$\frac{\partial^2 c_s(t)}{\partial s^2}=\frac{1}{4}\ddot{c}(t-s)+\frac{1}{4}\ddot{c}(t+s)=\frac{\partial^2 c_s(t)}{\partial t^2}-\frac{1}{2}\ddot{c}(t).$$

Then $\widetilde{c}_s(t):=c_s(t)-\frac{1}{2}c(t)=\left\{c(t-s)+c(t+s)\right\}/4$ defines a solution of the *wave equation:*

$$\frac{\partial^2\widetilde{c}_s(t)}{\partial s^2}=\frac{\partial^2\widetilde{c}_s(t)}{\partial t^2}$$

with initial conditions $\widetilde{c}_0(t)=c(t)/2$ and

$$\left.\frac{\partial\widetilde{c}_s(t)}{\partial s}\right|_{s=0}=0.$$

If the smooth curve c is a solution of Equation (17) then Proposition 3 implies:

$$\widetilde{c}_s(t)=(A(s)-\mathbb{1}/2)\left(\widetilde{c}_0(t)\right)+b(s)=\text{co}_B(s)\left(\widetilde{c}_0(t)\right)+b(s).$$

Hence during the evolution $s\mapsto\widetilde{c}_s$ the affine form of the curve $\widetilde{c}=\widetilde{c}_0$ is preserved. This motivates the notion *soliton* for these solutions of the wave equation.

In the sequel we study which invertible matrices D can be written in the form $(\mathbb{1}+\text{co}_B(s))/2$ for some real matrix B. Since for any invertible matrix U we have $\text{co}_{UBU^{-1}}(s)=U\text{co}_B(s)U^{-1}$ it is sufficient to check the possible Jordan normal forms of $\text{co}_B(s)$ for the different Jordan blocks B. It turns out that a large class of invertible matrices D can be written in this form.

Proposition 4 (THE IMAGE OF $(B,s)\mapsto(\mathbb{1}+\text{co}_B(s))/2$). *Consider the mapping $f:(B,s)\in M(n;\mathbb{R})\times\mathbb{R}\mapsto f(B,s):=(\mathbb{1}+\text{co}_B(s))/2$. An invertible matrix $D\in M(n;\mathbb{R})$ lies in the image of f if and only if the (real or complex) Jordan normal form of D satisfies the following: If $\lambda<0$ is a real and negative eigenvalue of D and if for some $m\geq 1$ the Jordan block $J_m(\lambda)$ occurs in the Jordan normal form, then the number of Jordan blocks $J_m(\lambda)$ in the Jordan decomposition of D is even.*

In particular a diagonizable matrix D can be written as: $D=f(B,s)=(\mathbb{1}+\text{co}_B(s))/2$ for some $(B,s)\in M(n;\mathbb{R})\times\mathbb{R}$ if and only if the eigenspaces of all real and negative eigenvalues are even-dimensional.

Proof. We compute for the possible complex Jordan normal forms J of a real matrix B the complex Jordan normal form of $\mathrm{co}_J(s)$ resp. $f(J, s)$. Since the matrix $D = (\mathbb{1} + \mathrm{co}_B(s))/2$ is supposed to be invertible we exclude -1 as eigenvalue of $\mathrm{co}_B(s)$.

(a) If $J = \lambda \cdot \mathbb{1}, \lambda \in \mathbb{R}$, then $f(J, s) = (1 + \cos_\lambda(s))/2 \cdot \mathbb{1}$. Note that $\{(1 + \cos_\lambda(s))/2\,;\, \lambda, s \in \mathbb{R}\} = \{x \in \mathbb{R}\,;\, x \geq 0\}$. Hence $f(J, s)$ is a diagonal matrix with non-negative real eigenvalues.

(b) Assume the matrix B contains a complex Jordan block $J = u \cdot \mathbb{1}_m, u \in \mathbb{C} \setminus \mathbb{R}$, then also $\overline{J} = \overline{u} \cdot \mathbb{1}$ is a (distinct) Jordan block, i.e. there is an even-dimensional invariant subspace on which the Jordan normal form is given by $J \oplus \overline{J} = u \cdot \mathbb{1}_m \oplus \overline{u} \cdot \mathbb{1}_m$. Then $f\left(J \oplus \overline{J}, s\right) = (1 + \cosh(ws))/2 \cdot \mathbb{1} \oplus (1 + \cosh(\overline{w}s))/2 \cdot \mathbb{1}$, here $w \in \mathbb{C}, u = w^2$. Note that $\cosh : \mathbb{C} \to \mathbb{C}$ is surjective. If $\cosh(ws) \in \mathbb{R}$ the corresponding eigenspace is even-dimensional.

We conclude from (a) and (b): A real and invertible matrix D which is diagonalizable over $\mathbb{C}$ can be written in the form $D = f(B, s)$ for a matrix B diagonalizable over $\mathbb{C}$ and a real number s if and only if it does not have a real and negative eigenvalue λ with an odd-dimensional eigenspace.

(c) Now assume that $J = J_m(u)$ with $m \geq 2$. For $u = 0$: we obtain for $N_m = J_m(0)$:

$$(24) \qquad \mathrm{co}_{N_m}(s) = \mathbb{1}_m + \frac{s^2}{2!} N_m + \ldots + \frac{s^{2m-2}}{(2m-2)!} N_m^{m-1}\,.$$

Note that the matrix N_m^k satisfies $N_m^k(e_{l+k}) = e_l, l = 1, 2, \ldots, m-k, N_m^k(e_l) = 0, l = 1, 2, \ldots, k-1$. Since for $s \neq 0$

$$\mathrm{rank}\left(\mathrm{co}_{N_m}(s) - \mathbb{1}_m\right) = m - 1$$

we obtain: For $s \neq 0$ the matrix $\mathrm{co}_{N_m}(s)$ is conjugate to the Jordan block $J_m(1)$ and $f(N_m, s)$ is also conjugate to the Jordan block $J_m(1)$.

For $u \neq 0$ we obtain:

$$(25) \quad \mathrm{co}_J(s) = \sum_{k=0}^{\infty} \frac{s^{2k}}{(2k)!} J^k = \sum_{k=0}^{\infty} \frac{s^{2k}}{(2k)!} \sum_{j=0}^{\min(m,k)} \binom{k}{j} u^{k-j} N_m^j$$

$$= \sum_{k=0}^{\infty} \frac{s^{2k}}{(2k)!} u^k \cdot \mathbb{1}_m + \frac{s}{2} \sum_{k=1}^{\infty} \frac{s^{2k-1}}{(2k-1)!} u^{k-1} \cdot N_m +$$

$$+ \alpha_2(s) N_m^2 + \ldots + \alpha_{m-1}(s) \cdot N_m^{m-1}$$

for some $\alpha_2(s), \ldots, \alpha_{m-1}(s) \in \mathbb{C}$. If $u = \lambda \in \mathbb{R}$ we have

$$\mathrm{co}_J(s) = \cos_\lambda(s) \cdot \mathbb{1}_m + \frac{s}{2} \sin_\lambda(s) \cdot N_m + \alpha_2 N_m^2 + \ldots + \alpha_{m-1} N_m^{m-1}\,.$$

Note that $\{\cos_\lambda(s)\,;\, \lambda, s \in \mathbb{R}\} = \{x \in \mathbb{R}, x \geq -1\}$ and that $\sin_\lambda(s) \neq 0$ whenever $\cos_\lambda(s) \neq \pm 1$. Hence for $\cos_\lambda(s) \neq \pm 1$ the matrix $\mathrm{co}_J(s)$ satisfies:

$$\mathrm{rank}\left(\mathrm{co}_J(s) - \cos_\lambda(s) \cdot \mathbb{1}_m\right) = m - 1$$

i.e. the matrix $f(J_m(\lambda), s)$ is conjugate to the Jordan block $J_m((1 + \cos_\lambda(s))/2)$. Thus we have shown that a Jordan block $J_m(\mu)$ with real $\mu \neq 0$ and $m \geq 2$ can be written in the form $f(J_m(\lambda), s)$ for some real λ and some $s \in \mathbb{R}$ up to conjugacy if and only if μ is positive.

(d) Now assume that the real matrix B contains a complex Jordan block $J = J_m(u), u \in \mathbb{C} \setminus \mathbb{R}$. Then let $w \in \mathbb{C} \setminus \mathbb{R}$ be a square root, i.e. $u = w^2$. Hence also $\overline{J} = J_m(\overline{u})$ is a (distinct) Jordan block of B, or rather there is an invariant subspace on which the Jordan normal form of A is given by $J \oplus \overline{J} = J_m(u) \oplus J_m(\overline{u})$. We conclude from Equation 25:

$$\mathrm{co}_J(s) = \cosh(ws) \cdot \mathbb{1}_m + \frac{s}{2w} \sinh(ws) N_m + \alpha_2(s) N_m^2 + \ldots + \alpha_{m-1}(s) \cdot N_m^{m-1}$$

for some $\alpha_2(s), \ldots, \alpha_{m-1}(s) \in \mathbb{C}$. If $\cosh(ws) \neq \pm 1$ we have $\sinh(ws) \neq 0$, i.e.

$$\mathrm{rank}\,(\mathrm{co}_J(s) - \cosh(ws) \cdot \mathbb{1}_m) = m - 1$$

or rather the matrix $\mathrm{co}_J(s)$ is conjugate to a Jordan block $J_m(\cosh(ws))$ and $f(J \oplus \overline{J})$ is conjugate to $J_m((1 + \cosh(ws))/2) \oplus J_m((1 + \cosh(\overline{w}s))/2)$. We conclude from (c) and (d) that an invertible and real matrix D whose Jordan normal form contains a Jordan block $J = J_m(v), m \geq 2$ can be written in the form $f(B, s)$ for a real matrix B and $s \in \mathbb{R}$ if and only if the following holds: If v is real and negative $v = \lambda < 0$ then the number of Jordan blocks $J_m(\lambda)$ in the Jordan decomposition of B is even. $\square$

3. Discrete Curve Shortening

An (infinite) *polygon* $x = (x_j)_{j \in \mathbb{Z}}$ in $\mathbb{R}^n$ is defined by its *vertices* $x_j \in \mathbb{R}^n$. We call $\mathcal{P} = \mathcal{P}(\mathbb{R}^n)$ the vector space of these polygons. We can identify the polygon x with the piecewise linear curve $x : \mathbb{R} \to \mathbb{R}^n$ which is a straight line on any interval $[j, j+1]$ and satisfies $x(j + u) = (1 - u)x_j + ux_{j+1}$ for any $u \in [0, 1], j \in \mathbb{Z}$. If there is a positive number N such that $x_{j+N} = x_j$ for all $j \in \mathbb{Z}$, then we call the polygon x *closed* or rather *periodic* with N vertices or rather of period N. In this case we can identify the index set with $\mathbb{Z}_N = \mathbb{Z}/(N \cdot \mathbb{Z})$. We denote the set of closed polygons with N vertices by $\mathcal{P}_N = \mathcal{P}_N(\mathbb{R}^n)$. The *midpoint mapping* is given by

$$M : x \in \mathcal{P}(\mathbb{R}^n) \longmapsto M(x) \in \mathcal{P}(\mathbb{R}^n)\,; \ (M(x))_j := \frac{1}{2}(x_j + x_{j+1})\,.$$

For a closed polygon $x \in \mathcal{P}_N(\mathbb{R}^n)$ its *length* is given by $L(x) := \sum_{j=0}^{N-1} \|x_{j+1} - x_j\|$, here $\|.\|$ denotes the Euclidean norm. The triangle inequality implies the following curve shortening property of the midpoint mapping in the general case:

$$\begin{aligned} L(M(x)\,|[j, j+m]) &= \sum_{k=1}^{m} \left\| (M(x))_{j+k} - (M(x))_{j+k-1} \right\| \\ &\leq L(x\,|[j + 1/2, j + m + 1/2]) \end{aligned}$$

with equality if and only if the points $x_j, x_{j+1}, \ldots, x_{j+m}, x_{j+m+1}$ lie on a straight line. Thus the midpoint mapping M is *length decreasing* on closed polygons, i.e. for all closed polygons x we have: $L(M(x)) \leq L(x)$ with equality if and only if x is constant, i.e. $L(x) = 0$. We modify the midpoint mapping as follows:

Definition 2 (CURVE SHORTENING PROCESS). *We introduce the following mapping* $T : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(\mathbb{R}^n)$:

$$(T(x))_j = \frac{1}{4}\left\{x_{j-1} + 2x_j + x_{j+1}\right\}.$$

(a) We call a polygon $x = (x_j)_{j\in\mathbb{Z}}$ affinely invariant *under the mapping* T *or rather an* (affine) soliton *of the curve shortening process* T*) if there is an affine map* $(A, b), A \in Gl(n, \mathbb{R}), b \in \mathbb{R}^n$ *such that for all* $j \in \mathbb{Z}$:

$$(26) \qquad T(x)_j = \frac{1}{4}\left\{x_{j-1} + 2x_j + x_{j+1}\right\} = A(x_j) + b.$$

(b) We call a smooth curve $c : \mathbb{R} \to \mathbb{R}^n$ affinely invariant *under the mapping* T *(or rather an* (affine) soliton *of the curve shortening process* T*) if there is some* $\epsilon > 0$ *such that there is a one-parameter family* $s \in (-\epsilon, \epsilon) \mapsto (A(s), b(s)) \in Gl(n, \mathbb{R}) \times \mathbb{R}^n$ *of affine maps such that*

$$(27) \qquad c_s(t) = \frac{1}{4}\left\{c(t-s) + 2c(t) + c(t+s)\right\} = A(s)c(t) + b(s)$$

for all $s \in (0, \epsilon), t \in \mathbb{R}$.

It is obvious that these notions are affinely invariant. For $a \in \mathbb{R}, s \in (0, \epsilon)$ the polygon $x = x(a, s)$ with $x(a, s)_j = c(a + sj), j \in \mathbb{Z}$ lying on a smooth curve $c : \mathbb{R} \longrightarrow \mathbb{R}^n$ is a soliton of the curve shortening process T if the curve c is also a soliton of the corresponding process T on curves. On the other hand: A polygon $x = (x_j)_{j\in\mathbb{Z}}$ which is a soliton of the curve shortening process T satisfying Equation (26) can be obtained from a smooth curve, which is a soliton as defined in Equation (27) if and only if A can be written in the form $A = (\mathbb{1} + \mathrm{co}_B(s))/2$. In Proposition 4 the Jordan normal form of these matrices are classified.

Remark 4 (EIGENPOLYGONS OF T). If we consider closed polygons then the midpoint mapping defines a linear map $M : \mathcal{P}_N \longrightarrow \mathcal{P}_N$ on the $(n \cdot N)$-dimensional vector space $\mathcal{P}_N$, and one can use a decomposition into eigenspaces, cf. [14] and [2]. The matrix is in particular *circulant*. For $n = 2$ one can identify the complex numbers $\mathbb{C}$ with $\mathbb{R}^2$, then the eigenvalues are $\lambda_k = (1 + \exp(2\pi i k/N))/2, k = 0, 1, \ldots, N-1$. The corresponding eigenvectors are the polygons

$$(28) \qquad z^{(k)} = \left(z_1^{(k)}, \ldots, z_N^{(k)}\right) \in \mathbb{C}^N \; ; \; z_j^{(k)} = \exp(2\pi i j k/N)$$

for $k = 0, 1, \ldots, N-1$. Then the linear map $T : \mathcal{P}_N \longrightarrow \mathcal{P}_N$ is also circulant and has the form

$$T = \frac{1}{4} \begin{pmatrix} 2 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & \ldots & 0 & 0 & 0 & 0 \\ & \ldots & & & & & & & \\ & \ldots & & & & & & & \\ & \ldots & & & & & & & \\ 0 & 0 & 0 & 0 & \ldots & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 2 & 1 \\ 1 & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & 2 \end{pmatrix}$$

with eigenvalues

$$\mu_k = \mu_{N-k} = \frac{1}{2} \left(1 + \cos \left(\frac{2\pi k}{N} \right) \right), k = 0, 1, \ldots, N-1$$

and eigenvectors $z^{(k)}$ given by Equation (28). Note that all polygons $z^{(k)}$ given by Equation (28) lie on the unit circle $c(t) = \exp(2\pi i t)$ and $z_j^{(k)} = c(kj/N)$. The eigenpolygons $z^{(k)}$ are solitons on which the map T is a homothety. Although the curve c is simple and convex the polygons $z^{(k)}$ form regular N-gons which are simple and convex only for $k = 1, N-1$, cf. [2, Fig.5].

Proposition 5 (ASSIGNMENT MATRIX $A \to$ POLYGON). *Let $(A, b) : x \in \mathbb{R}^n \longmapsto Ax + b \in \mathbb{R}^n$ be an affine map and $u, v \in \mathbb{R}^n$ be two points in $\mathbb{R}^n$, and $j_0 \in \mathbb{Z}$. Then there is a unique polygon $x \in \mathcal{P}(\mathbb{R}^n)$ with $x_{j_0} = u, x_{j_0+1} = v$ which is affinely invariant (with respect to A and b) under the mapping T.*

Proof. If $x \in \mathcal{P}(\mathbb{R}^n)$ is affinely invariant under the mapping T we have

$$(29) \qquad (T(x))_j = \frac{1}{4} \{x_{j-1} + 2x_j + x_{j+1}\} = A(x_j) + b.$$

Hence the sequence (x_j) with $x_{j_0} = u, x_{j_0+1} = v$ is uniquely determined by the recursion formulae

$$(30) \; x_{j+1} = 2(2A - \mathbb{1})(x_j) - x_{j-1} + 4b; \; x_{j-1} = 2(2A - \mathbb{1})(x_j) - x_{j+1} + 4b.$$

$\square$

For a given smooth curve $c : \mathbb{R} \to \mathbb{R}^n$ we define the one-parameter family $c_s : \mathbb{R} \to \mathbb{R}^n$ by Equation (5). The curves c_s are obtained from $c = c_0$ by applying the mapping T as follows. For arbitrary $a \in \mathbb{R}, s > 0$ let $x = x(a, s)$ be the polygon $x_j = x_j(a, s) := c(a + js), j \in \mathbb{Z}$. Then

$$c_s(a + js) = (T(x))_j = \frac{1}{4} \{c(a + (j-1)s) + 2c(a + js) + c(a + (j+1)s)\}.$$

Hence the vertices $(T(x(a, s)))_j$ of the image $T(x(a, s))$ of the polygon $x = x(a, s)$ under the mapping T lie on the curve c_s, or rather the curve c_s is

formed by the images $T\left(x\left(a,s\right)\right)$ of polygons of the form $x=x(a,s)$ on the curve c.

Theorem 1 (SOLITONS AS SOLUTIONS OF AN ODE). *Let $c : \mathbb{R} \to \mathbb{R}^n$ be a smooth curve such that the one-parameter family defined by Equation (5) defined for all $s \in (-\epsilon, \epsilon)$ and some $\epsilon > 0$ satisfies:*

$$c_s(t) = A(s)(c(t)) + b(s)$$

for all $t \in \mathbb{R}, -\epsilon < s < \epsilon$ and for a smooth one-parameter family $s \mapsto A(s) \in Gl(n, \mathbb{R})$ of linear isomorphisms and a smooth curve $s \mapsto b(s) \in \mathbb{R}^n$. I.e. the curve c is affinely invariant *under the mapping T, cf. Definition 2 (b). Assume in addition that for some $t_0 \in \mathbb{R}$ the vectors $\dot{c}(t_0), \ddot{c}(t_0), \ldots, c^{(n)}(t_0)$ are linearly independent.*
Then the curve c is the unique solution of the differential equation

$$\text{(31)} \qquad\qquad \ddot{c}(t) = Bc(t) + d$$

with initial conditions $v = c(0), w = \dot{c}(0)$. The functions $s \mapsto A(s) \in M(n, \mathbb{R}), s \mapsto b(s) \in \mathbb{R}^n$ satisfy the differential equations

$$\text{(32)} \qquad A''(s) = (A(s) - \mathbb{1}/2)\, B\,;\; b''(s) = (A(s) - \mathbb{1}/2)\, d$$

with $B = 2A''(0), d = 2b''(0)$ and initial conditions $A(0) = \mathbb{1}, b(0) = 0, A'(0) = 0, b'(0) = 0$. In particular, we have $A(s) = (\mathbb{1} + \mathrm{co}_B(s))/2$, cf. Proposition 3. The possible Jordan normal forms of the matrices $A(s)$ are given in Proposition 4.

For a one-parameter family $s \mapsto c_s$ of curves or a one-parameter family $s \mapsto A(s), s \mapsto b(s)$ of matrices $A(s)$ or vectors $b(s)$ we denote differentiation with respect to s by $'$. Differentiation with respect to the curve parameter t of a one-parameter family of curves $c_s(t)$ or a single curve $t \mapsto c(t)$ is denoted by $\dot{c}$.
Note that for the *potential*

$$\text{(33)} \qquad\qquad U(x) = -\frac{1}{2}\langle Bx, x \rangle - \langle d, x \rangle$$

we can write instead of Equation (31) the following Equation:

$$\text{(34)} \qquad\qquad \ddot{c}(t) = -\mathrm{grad}\, U\left(c(t)\right)\,.$$

It follows that the function

$$f(t) := \|\dot{c}(t)\|^2 - \frac{1}{2}\langle Bc(t), c(t) \rangle - \langle d, c(t) \rangle$$

is constant for any solution of Equation (31).

Proof. Let

$$\text{(35)} \qquad c_s(t) = A(s)c(t) + b(s) = \frac{1}{4}\left\{c(t-s) + 2c(t) + c(t+s)\right\}\,.$$

For $s = 0$ we obtain $c(t) = c_0(t) = A(0)c(t) + b(0)$ for all $t \in \mathbb{R}$, or rather $(A(0) - \mathbb{1})(c(t)) = -b(0)$ for all t. We conclude that

$$(36) \qquad (A(0) - \mathbb{1})\left(c^{(k)}(t)\right) = 0$$

for all $k \geq 1$. Since for some t_0 the vectors $\dot{c}(t_0), \ddot{c}(t_0), \ldots, c^{(n)}(t_0)$ are linearly independent by assumption we conclude from Equation (36): $A(0) = \mathbb{1}, b(0) = 0$. Equation (35) implies for $k \geq 1$:

$$A(s)c^{(k)}(t) = \frac{1}{4}\left\{c^{(k)}(t-s) + 2c^{(k)}(t) + c^{(k)}(t+s)\right\}.$$

and hence

$$A'(s)c^{(k)}(t) = -\frac{1}{4}\left\{c^{(k+1)}(t-s) - c^{(k+1)}(t+s)\right\}$$

or rather

$$A'(0)c^{(k)}(t) = 0.$$

Since for some t_0 the vectors $\dot{c}(t_0), \ldots, c^{(n)}(t_0)$ are linearly independent we obtain $A'(0) = 0$ and hence by Equation (35): $b'(0) = 0$.
We conclude from Equation (35):

$$\begin{aligned}
\frac{\partial^2 c_s(t)}{\partial s^2} &= A''(s)c(t) + b''(s) \\
&= \frac{\partial^2 c_s(t)}{\partial t^2} - \frac{1}{2}\ddot{c}(t) = (A(s) - \mathbb{1}/2)\ddot{c}(t).
\end{aligned}$$

Since $A(0) = \mathbb{1}$ the endomorphisms $A(s) - \mathbb{1}/2$ are isomorphisms for all $s \in (0, \epsilon)$ for a sufficiently small $\epsilon > 0$. Hence we obtain for $s \in (0, \epsilon)$:

$$(37) \qquad \ddot{c}(t) = (A(s) - \mathbb{1}/2)^{-1} A''(s)c(t) + (A(s) - \mathbb{1}/2)^{-1} b''(s).$$

Differentiating with respect to s:

$$\left((A(s) - \mathbb{1}/2)^{-1} A''(s)\right)'(c(t)) + \left((A(s) - \mathbb{1}/2)^{-1} b''(s)\right)' = 0$$

and differentiating with respect to t:

$$\left((A(s) - \mathbb{1}/2)^{-1} A''(s)\right)' c^{(k)}(t) = 0\,; \; k = 0, 1, 2, \ldots, n.$$

Since $\dot{c}(t_0), \ddot{c}(t_0), \ldots, c^{(n)}(t_0)$ are linearly independent by assumption we conclude that

$$\left((A(s) - \mathbb{1}/2)^{-1} A''(s)\right)' = 0.$$

Hence with $B = 2A''(0), d = 2b''(0)$ we obtain

$$A''(s) = (A(s) - \mathbb{1}/2) B\,; \; b''(s) = (A(s) - \mathbb{1}/2)(d),$$

i.e. the differential Equations (32). For $s = 0$ we obtain from Equation (37):

$$\ddot{c}(t) = Bc(t) + d\,,$$

i.e. Equation (31). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

We can combine the results of Theorem 1 and Proposition 3 to give the following characterization of affinely invariant curves under the affine mapping T as solutions of an inhomogeneous linear differential equation of second order:

Theorem 2 (CHARACTERIZATION OF AFFINELY INVARIANT CURVES AS SOLUTIONS OF AN ODE). *(a) Let $c : \mathbb{R} \longrightarrow \mathbb{R}^n$ be a smooth curve affinely invariant under the mapping T. Assume in addition that for some $t_0 \in \mathbb{R}$ the vectors $\dot{c}(t_0), \ddot{c}(t_0), \ldots, c^{(n)}(t_0)$ are linearly independent. Then there is a unique matrix $B \in M(n; \mathbb{R})$ and a unique vector $d \in \mathbb{R}^n$ such that*

$$\ddot{c}(t) = Bc(t) + d.$$

(b) Let B be a real matrix and d a vector in $\mathbb{R}^n$. Then any solution $c = c(t)$ of the inhomogenous linear differential equation $\ddot{c}(t) = Bc(t) + d$ with constant coefficients defines an affinely invariant smooth curve under the mapping T.

For a closed polygon $x \in \mathcal{P}_N$ the *center of mass* x_{cm} is given by

(38)
$$x_{cm} = \frac{1}{N} \sum_{j=1}^{N} x_j \,.$$

Since $(T(x))_{cm} = x_{cm}$ we conclude: There is no translation-invariant closed polygon, since the center of mass is preserved under the curve shortening processes B or rather T.

Remark 5 (GENERALIZATION OF THE MAP T). For three points $x, y, z \in \mathbb{R}^n$ define the affine map

$$\mathcal{T} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n \,;\, \mathcal{T}(x, y, z) = \frac{1}{4} \left\{ x + 2y + z \right\} \,.$$

Hence the mapping $T : \mathcal{P}(\mathbb{R}^n) \longrightarrow \mathcal{P}(\mathbb{R}^n)$ introduced in Definition 2 satisfies for all $j \in \mathbb{Z}$:

$$T(x)_j = \mathcal{T}\left(x_{j-1}, x_j, x_{j+1}\right) \,.$$

The one-parameter family c_s associated to the smooth curve c by Equation (5) can be written as:

$$c_s(t) := \mathcal{T}\left(c(t-s), c(t), c(t+s)\right) = \frac{1}{4} \left\{ c(t-s) + 2c(t) + c(t+s) \right\} \,.$$

In the following we will allow a slightly more general curve shortening process

(39) $$T_\alpha : \mathcal{P}(\mathbb{R}^n) \longrightarrow \mathcal{P}(\mathbb{R}^n), (T_\alpha(x))_j := \mathcal{T}_\alpha\left(x_{j-1}, x_j, x_{j+1}\right)$$

based on the affine map $\mathcal{T}_\alpha : \mathbb{R}^n \longrightarrow \mathbb{R}^n \,;\, \mathcal{T}_\alpha(x, y, z) = \alpha x + (1 - 2\alpha)y + \alpha z$ for $\alpha \neq 0$, i.e. $\mathcal{T} = \mathcal{T}_{1/4}$. For $\alpha = 1/3$ the point $\mathcal{T}_{1/3}(x, y, z)$ is the *center of mass* $(x + y + z)/3$. The curve $\alpha \in [0, 1/2] \mapsto \mathcal{T}_\alpha\left(x_{j-1}, x_j, x_{j+1}\right) \in \mathbb{R}^n$ is a parametrization of the straight line connecting x_j with the midpoint $(x_{j-1} + x_{j+1})/2$ of the points x_{j-1}, x_{j+1}. These mappings T_α are considered

for example in [2, p.238-39] and [3, ch.5.1]. For a smooth curve c one defines the associated one-parameter family of curves

$$c_{\alpha,s}(t) = \mathcal{T}_\alpha\left(x(t-s), c(t), c(t+s)\right) = \alpha c(t-s) + (1-2\alpha)\,c(t) + \alpha c(t+s)\,.$$

We call a smooth curve c *affinely invariant* (or a *soliton*) with respect to T_α if there is a one-parameter family $(A_\alpha(s), b_\alpha(s))\,, s \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$ of affine mappings such that $c_{\alpha,s}(t) = A_\alpha(s)c(t) + b_\alpha(s)$ for all $t \in \mathbb{R}, s \in (-\epsilon, \epsilon)$. Then

$$c_s(t) = c_{1/4,s}(t) = c(t) + \frac{1}{4}\left\{c(t-s) - 2c(t) + c(t+s)\right\}$$
$$= \frac{(4\alpha - 1)c(t) + c_{\alpha,s}(t)}{4\alpha}$$

and

$$c_{\alpha,s}(t) = 4\alpha c_s(t) - (4\alpha - 1)c(t)\,.$$

We conclude: A smooth curve c is a soliton for the transformation T_α for $\alpha \neq 0$ if and only if it is a soliton for the transformation $T = T_{1/4}$ and for the corresponding affine maps $(A_\alpha(s), b_\alpha(s))$ we obtain: $A_\alpha(s) = 4\alpha A(s) - (4\alpha - 1)\mathbb{1}\,; b_\alpha(s) = 4\alpha b(s)\,.$

4. Semidiscrete flows of polygons

The mapping T introduced in Definition 2 or T_α defined in Remark 5 or Equation (39) can be seen as a discrete version of the *semidiscrete flow* defined on the space $\mathcal{P}(\mathbb{R}^n)$ of polygons: For a given polygon $(x_j)_{j\in\mathbb{Z}}$ the flow $s \mapsto (x_j(s)) \in \mathcal{P}(\mathbb{R}^n)$ is defined by the Equation

$$(40) \qquad \frac{dx_j(s)}{ds} = x_{j-1}(s) - 2x_j(s) + x_{j+1}(s)\,, x_j(0) = x_j\,.$$

This flow is discussed for example in [5]. It is a linear first order system of differential equations with constant coefficients forming a circulant matrix. Hence one can write down the solutions explicitly. If we approximate the left hand side of this equation by $(x_j(s+\alpha) - x_j(s))/\alpha$ we obtain

$$x_j(s+\alpha) = \alpha x_{j-1}(s) + (1-2\alpha)\,x_j(s) + \alpha x_{j+1}(s)$$
$$= \mathcal{T}_\alpha\left(x_{j-1}(s), x_j(s), x_{j+1}(s)\right)\,.$$

Therefore the mappings T and T_α can be seen as discrete versions of the flow Equation (40). In [5] the flow $s \mapsto x_j(s)$ introduced in Equation (40) is called *semidiscrete* since it is a smooth flow on a space of discrete objects (polygons). Then a discretization of the semidiscrete flow yields to the discrete process T and T_α discussed here. On the other hand the connection of the semidiscrete flow with the *smooth* curve shortening flow in Euclidean space is discussed in detail in [5, Sec. 5].

If we consider the functional

$$F_2 : \mathcal{P}_N \longrightarrow \mathbb{R}\,; F_2(x) = \frac{1}{2}\sum_{j=0}^{N} \|x_{j+1} - x_j\|^2$$

on the space $\mathcal{P}_N$ of closed polygons then we obtain for a curve $s \in (-\epsilon, \epsilon) \mapsto x(s) = (x_j(s))_{j \in \mathbb{Z}_N}$ with $x = x(0)$ and $\dot{x} = \dot{x}(0)$:

$$\left.\frac{dF_2(x(s))}{ds}\right|_{s=0} = -\sum_{j=0}^{N} \langle \dot{x}_j, x_{j-1} - 2x_j + x_{j+1} \rangle$$

and we obtain for the gradient $\mathrm{grad}F_2(x)$:

$$\mathrm{grad}F_2(x) = -\left(x_{j-1} - 2x_j + x_{j+1}\right)_{j \in \mathbb{Z}_N} .$$

Hence the semidiscrete flow can be viewed as the negative gradient flow of the functional F_2, cf. [5, Sec.6]. An affine transformation $x \in \mathbb{R}^n \mapsto \mathcal{A}(x) = A(x) + b \in \mathbb{R}^n$ on $\mathbb{R}^n$ induces an affine transformation $\widehat{\mathcal{A}}$ on $\mathcal{P}_N$: $\widehat{\mathcal{A}} = (x_1, \dots, x_n) = (\mathcal{A}(x_1), \dots, \mathcal{A}(x_N))$. In contrast to the functional F_2 its gradient $\mathrm{grad}F_2$ is invariant under $\widehat{\mathcal{A}}$:

$$\mathrm{grad}F_2\left(\widehat{\mathcal{A}}(x)\right) = -\left(A(x_{j-1}) - 2A(x_j) + A(x_{j+1})\right)_{j \in \mathbb{Z}_N} = \widehat{\mathcal{A}}\left(\mathrm{grad}F_2(x)\right) .$$

Definition 3 (AFFINE INVARIANCE UNDER THE SEMIDISCRETE FLOW). *We call a smooth curve* $c : \mathbb{R} \to \mathbb{R}^n$ *affinely invariant* under the semidiscrete flow *given by Equation* (40) *(or a* soliton*) if there is some* $\epsilon > 0$ *such that for any* $s \in (0, \epsilon)$ *there is an affine map* $\left(\widetilde{A}(s), \widetilde{b}(s)\right)$ *such that the one-parameter family*

(41) $$\widetilde{c}_s(t) = \widetilde{A}(s)c(t) + \widetilde{b}(s)$$

is a solution of the flow equation

$$\frac{\partial \widetilde{c}_s(t)}{\partial s} = \widetilde{c}_s(t-1) - 2\widetilde{c}_s(t) + \widetilde{c}_s(t+1)$$

for all $s \in (0, \epsilon), t \in \mathbb{R}$.

In the following Proposition we show that the solitons of the mapping T or T_α coincide with the solitons of the semidiscrete flow given by Equation (40):

Proposition 6 (AFFINELY INVARIANT CURVES UNDER THE SEMIDISCRETE FLOW). *Let B be a matrix and d a vector in $\mathbb{R}^n$ and let $c = c(t)$ be a solution of the inhomogenous linear differential equation $\ddot{c}(t) = Bc(t) + d$ with constant coefficients for which $\dot{c}(0), \ddot{c}(0), \dots, c^{(n)}(0)$ are linearly independent. Then the curve c defines an affinely invariant smooth curve under the semidiscrete flow given in Equation* (40).

Proof. We conclude from Proposition 3 and Equation (18) that there is a matrix $A_1 = (A(1) - \mathbb{1})/4$ and a vector $b_1 = b(1)/4$ such that

(42) $$c(t-1) - 2c(t) + c(t+1) = A_1 c(t) + b_1$$

for all $t \in \mathbb{R}$. Let

$$\widetilde{A}(s) = \exp(A_1 \cdot s) \, ; \, \widetilde{b}(s) = \int_0^s \exp(A_1 \cdot \sigma)(b_1) \, d\sigma \, ,$$
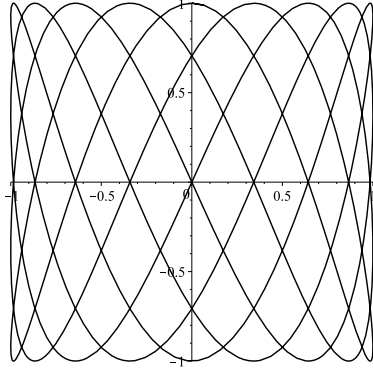
FIGURE 3. Scaling, Case (1a)
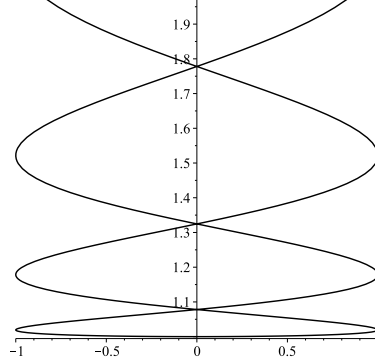
FIGURE 4. Scaling, Case (1b)

and

$$(43) \qquad \widetilde{c}_s(t) = \widetilde{A}(s)c(t) + \widetilde{b}(s).$$

Using $A_1 \cdot \widetilde{A}(s) = \widetilde{A}(s) \cdot A_1$ we obtain:

$$\frac{\partial \widetilde{c}_s(t)}{\partial s} = \widetilde{A}'(s)c(t) + \widetilde{b}'(s) = A_1 \cdot \widetilde{A}(s)c(t) + \widetilde{A}(s)(b_1)$$
$$= \widetilde{A}(s)\left(A_1 c(t) + b_1\right).$$

We conclude from Equation (42) and Equation (43):

$$\frac{\partial \widetilde{c}_s(t)}{\partial s} = \widetilde{A}(s)\left(c(t-1) - 2c(t) + c(t+1)\right) = \widetilde{c}_s(t-1) - 2\widetilde{c}_s(t) + \widetilde{c}_s(t+1).$$

Hence the curve c is affinely invariant under the semidiscrete flow. $\qquad \square$

5. Planar solitons

We study the planar case $n = 2$. We conclude from Theorem 2 that solitons c are solutions $c(t) = (x(t), y(t))$ of the differential equation

$$\ddot{c}(t) = Bc(t) + d.$$

We discuss these solutions using Proposition 2 and Proposition 3.
Let $B \in M(2; \mathbb{R})$ be a matrix in (real) Jordan normal form. Then we consider the following cases:

(1) B is diagonalizable (over $\mathbb{R}$) and invertible, and $d = 0$. i.e.

$$B = \left(\begin{array}{cc} b_1 & 0 \\ 0 & b_2 \end{array} \right), \ b_1, b_2 \in \mathbb{R} - \{0\},$$

this affine map is called *scaling*. If the diagonal entries coincide the transformation is also called a *homothety*. The differential equation $\ddot{c} = Bc$
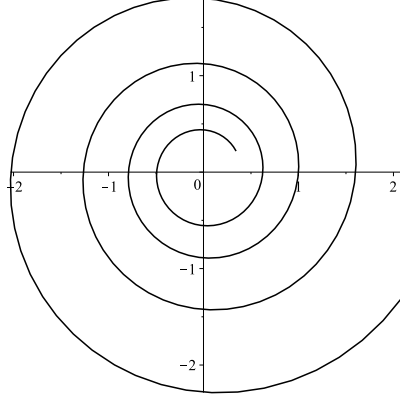
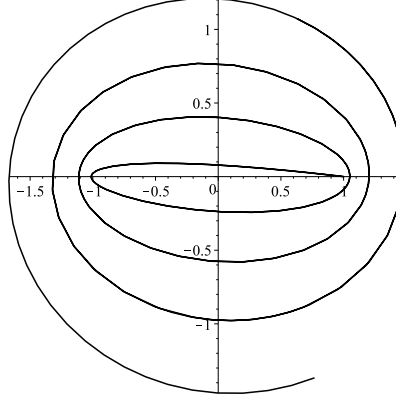FIGURE 5. *Spira mirabilis,* Rotate and scale, Case (2a)

FIGURE 6. Rotate and scale, Case (2b)

implies $\ddot{x} = b_1 x, \ddot{y} = b_2 y$. With the notation introduced in Equation (9) we obtain with $c(0) = (v_1, v_2), \dot{c}(0) = (w_1, w_2)$:

$$c(t) = (v_1 \cos_{b_1}(t) + w_1 \sin_{b_1}(t), v_2 \cos_{b_2}(t) + w_2 \sin_{b_2}(t))$$

The matrices $A(s)$ are also diagonal matrices (or scalings)

$$A(s) = \frac{1}{2} \{ \mathbb{1} + \operatorname{co}_B(s) \} = \frac{1}{2} \begin{pmatrix} 1 + \cos_{b_1}(s) & 0 \\ 0 & 1 + \cos_{b_2}(s) \end{pmatrix}.$$

In particular the diagonal entries of the matrices $A(s)$ are non-negative for all s. Hence the one parameter family c_s of curves is produced by *scaling* (with a diagonal matrix) from the soliton $c = c_0$.

Particular examples are:

(1a) We obtain *closed curves* if $b_1 = -\lambda_1^2 < 0, b_2 = -\lambda_2^2 < 0$ and $\lambda_1/\lambda_2 \in \mathbb{Q}$. Then we obtain for example *Lissajous* curves of the form

$$c(t) = (w_1 \sin(\lambda_1 t), v_2 \cos(\lambda_2 t))$$

see Figure 3 for $\lambda_1 = 4, \lambda_2 = 9, w_1 = v_2 = 1, w_2 = v_1 = 0$ and $0 \le t \le 6.3$. Another example is given in the Introduction, it is the curve $c(t) = (\cos(2t), \cos(3t))$. In Figure 2 we show the curve $c = c(t)$ and $c_{0.4} = c_{0.4}(t) = A(0.4) c(t)$:

$$c_{0.4}(t) = \frac{1}{2} \left((1 + \cos(0.8)) \cos(2t), (1 + \cos(1.2)) \cos(3t) \right).$$

(1b) If $b_1 = \lambda_1^2 > 0, b_2 = -\lambda_2^2 < 0$ we obtain for example curves of the following form:

$$c(t) = (w_1 \sin(\lambda_1 t), v_2 \cosh(\lambda_2 t))$$

In Figure 4 this curve is shown for the parameters $\lambda_1 = 8, \lambda_2 = 1, v_2 = w_1 = 1, v_1 = w_2 = 0$ and $-1.3 \le t \le 1.3$.

(1c) If B is diagonalizable and has the form

$$A = \begin{pmatrix} 0 & 0 \\ 0 & b_2 \end{pmatrix}, \ b_2 \neq 0.$$

and $d = 0$ then $\ddot{x}(t) = 0, \ddot{y} = b_2 y$, i.e. $c(t) = (t, v_2 \cos_{b_2}(t) + w_2 \sin_{b_2}(t))$ are solutions, for example

$$c(t) = (t, \exp(t)), c(t) = (t, \cosh(t)), c(t) = (t, \sin(t)).$$

(2) The following case corresponds to the case that the matrix B has no real eigenvalues. Hence B is a *similarity,* i.e. a composition of a *rotation* and a *dilation* $x \mapsto \lambda x$ for some $\lambda \neq 0$. We identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$ and assume that the matrix A is complex linear, i.e. can be identified with the multiplication with a non-zero complex number μ Then we are looking for a solution $z : t \in \mathbb{R} \mapsto z(t) = x(t) + iy(t) \in \mathbb{C}$ of the differential equation $\ddot{z} = \mu z$. For a complex number w with $\mu = w^2$, $w = u_1 + iu_2, u_1, u_2 \in \mathbb{R}, u_1 = \Re(w), u_2 = \Im(w)$ a solution has the form

$$z(t) = h_1 \exp(wt) + h_2 \exp(-wt).$$

for $h_1, h_2 \in \mathbb{C}$. If we write $h_1 = h_{11} + ih_{12}, h_2 = h_{21} + ih_{22}$ with $h_{11}, h_{12}, h_{21}, h_{22} \in \mathbb{R}$ then we obtain for $x(t) = \Re(z(t)), y(t) = \Im(z(t))$:

$$\begin{aligned} x(t) &= \{h_{11} \exp(u_1 t) + h_{21} \exp(-u_1 t)\} \cos(u_2 t) \\ &\quad + \{-h_{12} \exp(u_1 t) + h_{22} \exp(-u_1 t)\} \sin(u_2 t) \\ y(t) &= \{h_{12} \exp(u_1 t) + h_{22} \exp(-u_1 t)\} \cos(u_2 t) \\ &\quad + \{h_{11} \exp(u_1 t) - h_{21} \exp(-u_1 t)\} \sin(u_2 t) \end{aligned}$$

(44)

For arbitrary h_1, h_2 the one-parameter family $z_s(t)$ in complex notation is given by

$$\begin{aligned} z_s(t) &= \frac{1}{4} \{z(t-s) + 2z(t) + z(t+s)\} \\ &= \frac{1}{2} (1 + \cosh(sw)) (h_1 \exp(wt) + h_2 \exp(-wt)). \end{aligned}$$

In real notation we obtain the matrix:

$$A(s) = \frac{1}{2} \begin{pmatrix} 1 + \cosh(u_1 s) \cos(u_2 s) & -\sinh(u_1 s) \sin(u_2 s) \\ \sinh(u_1 s) \sin(u_2 s) & 1 + \cosh(u_1 s) \cos(u_2 s). \end{pmatrix}$$

Particular examples are:

(2a) The logarithmic spiral (*spira mirabilis*):

$$c(t) = \exp(u_1 t) (\cos(u_2 t), \sin(u_2 t)).$$

i.e. $h_{11} = 1, h_{12} = h_{21} = h_{22} = 0$. Figure 5 shows this curve for $u_1 = 0.3, u_2 = 4$ and $-3 \leq t \leq 3$.
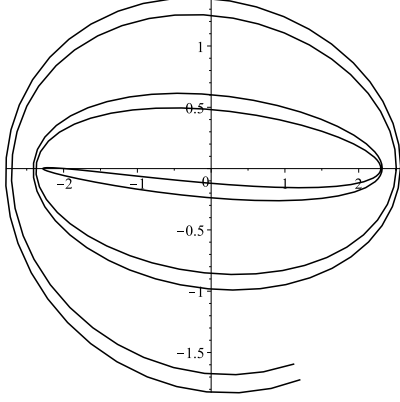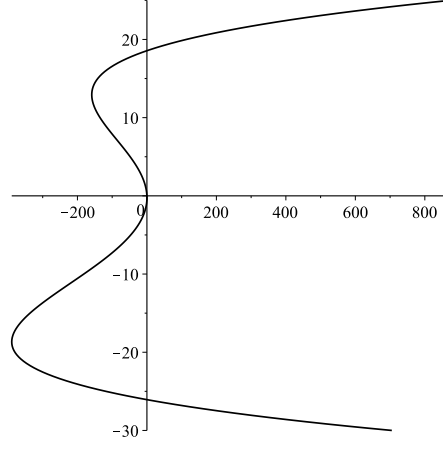
FIGURE   7. Rotate
and scale, Case (2c)

FIGURE   8. Shear,
Case (3)

(2b) The curve

$$c(t) = (\cosh(u_1 t) \cos(u_2 t), \sinh(u_1 t) \sin(u_2 t))$$

(i.e. $h_{11} = h_{21} = 1/2, h_{12} = h_{22} = 0$) is shown in Figure 6 for the values $u_1 = 1, u_2 = 20$ and $0 \le t \le 1.2$.

(2c) The curve given in Equation (44) for $h_{11} = 1, h_{21} = 1.3, h_{12} = h_{22} = 0, u_1 = 1, u_2 = 20$ is shown in Figure 7 for $-0.57 \le t \le 0.885$.

(3) Let B be non-zero and nilpotent, i.e.

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $d = (0, d_2)$.

Then a solution is given by $c(t) = \left(d_2 t^4/24 + a_3 t^3/6 + a_2 t^2/2 + a_1 t + a_0, t\right)$ for $a_0, a_1, a_2, a_3 \in \mathbb{R}$. The matrices $A(s)$ are of the form:

$$A(s) = \begin{pmatrix} 1 & s^2/2 \\ 0 & 1 \end{pmatrix},$$

cf. the Proof of Proposition 4.Hence the one-parameter family $s \mapsto c_s$ is formed by *shear transformations*. The curve with parameters $d_2 = 0.1, a_3 = 0.2, a_2 = -4, a_1 = -1, a_0 = 0$ for $-30 \le t \le 25$ is shown in Figure 8.

(4) Let B be invertible with real eigenvalue and not diagonizable, i.e.

$$B = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$$

with $b \in \mathbb{R}, b \ne 0$ and $d = 0$. Then $c(t) = (x(t), y(t))$ with

$$x(t) = \left(v_1 + \frac{w_2}{2b}t\right) \cos_b(t) + \left(\frac{w_1}{b} - \frac{w_2}{2b^2} + \frac{v_2}{2}t\right) \sin_b(t)$$
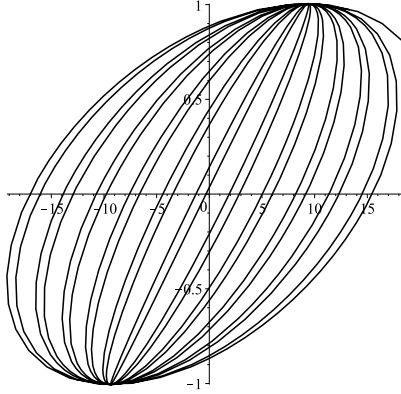$$y(t) = v_2\cos_b(t) + w_2\sin_b(t)$$
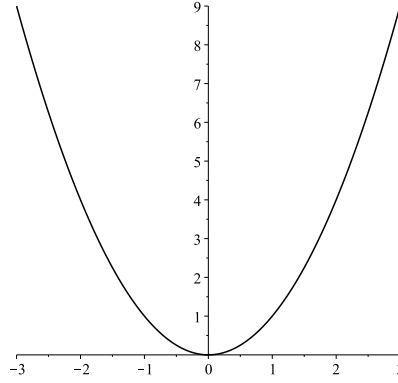
FIGURE 9. Shear and Scaling, Case (4)

FIGURE 10. Translation, Case (5)
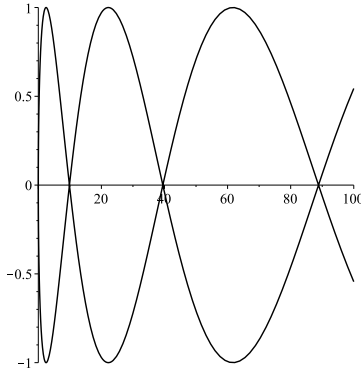

FIGURE 11. Translation and Scaling, Case (6)

The matrices $A(s)$ are of the form
$$A(s) = \frac{1}{2}\begin{pmatrix} 1+\cos_b(s) & s\cdot\sin_b(s)/2 \\ 0 & 1+\cos_b(s) \end{pmatrix},$$
cf. the Proof of Proposition 4. Hence the one-parameter family $s \mapsto c_s$ is formed by a composition of *shear transformations* and *scalings*. For the parameters $b=-1, v_1=1, v_2=-0.1, w_1=-10, w_2=1$ and $-30 \le t \le 40$ the curve is shown in Figure 9.

(5) If $B=0$ and $d \neq 0$, then we obtain (up to an affine transformations) the parabola $c(t)=(t,t^2)$ as translation-invariant curve, cf. Proposition 3(c) and Figure 10.

(6) If B is of the form
$$B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}$$

with non-zero b and $b = (2, 0)$. Then $\ddot{x}_1(t) = 2$ and $\ddot{x}_2(t) = bx_2(t)$. Then

$$c(t) = \left(t^2, v_1 \cos_b(t) + w_1 \sin_b(t)\right)$$

Examples are (for $b = \pm\lambda^2$:)

$$c(t) = \left(t^2, \cos(\lambda t)\right), c(t) = \left(t^2, \exp(\lambda t)\right), c(t) = \left(t^2, \cosh(\lambda t)\right) \ .$$

In Figure 11 the curve $c(t) = (t^2, \sin t), -10 \leq t \leq 10$ is shown.

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